STEENROD SQUARES ON INTERSECTION COHOMOLOGY AND A CONJECTURE OF M. GORESKY AND W. PARDON

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ABSTRACT. We prove a conjecture raised by M. Goresky and W. Pardon, concerning the range of validity of the perverse degree of Steenrod squares in intersection cohomology. This answer turns out of importance for the definition of characteristic classes in the framework of intersection cohomology.

For this purpose, we present a construction of cup_i-products on the cochain complex of filtered face sets, built on the blow-up of simplices and introduced in a previous work. We extend to this setting the classical properties of the associated Steenrod squares, including Adem and Cartan relations, for any generalized perversities. In the case that the filtered face set is the singular filtered face set associated to a pseudomanifold, we prove that our definition coincides with M. Goresky's definition.

Several examples of concrete computation of perverse Steenrod squares are given, including the case of isolated singularities and, more especially, we describe the Steenrod squares on the Thom space of a vector bundle, in function of the Steenrod squares of the basis and the Stiefel-Whitney classes. We detail also an example of a non trivial square, $\mathrm{Sq}^2\colon H_{\overline{p}} \to H_{\overline{p}+2}$, whose information is lost if we consider it as values in $H_{2\overline{p}}$, showing the interest of the Goresky and Pardon's conjecture.

Intersection cohomology was introduced by Goresky and MacPherson in [8] and [9], in order to adapt Poincaré duality to the case of singular manifolds and extend characteristic classes to this paradigm. Steenrod squares on the intersection cohomology of a pseudomanifold, X, have already be defined and studied by M. Goresky in [7]. For that, he uses a sheaf introduced by Deligne and proves that the Steenrod construction of cup_i-products induces a morphism $\mathrm{Sq}^i_G\colon H^r_{\overline{p}}(X;\mathbb{Z}_2) \to H^{r+i}_{2\overline{p}}(X;\mathbb{Z}_2)$, for any Goresky-MacPherson perversity $\overline{p}$ such that $2\overline{p}(\ell) \leq \ell - 1$ for any ℓ.

Here, we consider the extension of intersection cohomology to a simplicial setting previously introduced in [4]. The basic objects of study are the filtered face sets (see Definition 1.1), the case of a pseudomanifold, X, corresponding to a filtered version, $\underline{\mathrm{ISing}}^{\mathcal{F}}(X)$, of the singular simplicial set associated to a topological space. As cochain complex associated to a filtered face set, $\underline{K}$, we choose a blow-up $\widetilde{N}^*(\underline{K})$ of the normalized cochain complex on a simplicial set. This notion of blow-up, defined in [4] and recalled in Section 1, comes from a version in differential forms already existent in [3]. The elements of $\widetilde{N}^*(\underline{K})$ have a perverse degree (see Definition 1.3) which allows the definition of a complex $\widetilde{N}^*_{\overline{p}}(\underline{K})$, for any loose perversity $\overline{p}$. In the case of the singular filtered

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face set, $\underline{\mathrm{ISing}}^{\mathcal{F}}(X)$, we have proved in [4] that $\widetilde{N}^*_{\overline{p}}(\underline{K})$ gives the Goresky-MacPherson cohomology of the pseudomanifold X.

When the coefficients of $\widetilde{N}^*(\underline{K})$ are in a field of characteristic 2, $\mathbb{F}_2$, we define a structure of cup$_i$-products, $\cup_i\colon \widetilde{N}^*_{\overline{p}}(\underline{K})\otimes\widetilde{N}^*_{\overline{q}}(\underline{K}) \to \widetilde{N}^*_{\overline{p}+\overline{q}}(\underline{K})$, for any loose perversities $\overline{p}$, $\overline{q}$. This is done from the work of C. Berger and B. Fresse in [1] (see also [14]): we consider a normalized, homogeneous Bar resolution, $\mathcal{E}(2)$, of the symmetric group Σ_2 and prove that there exists a Σ_2-equivariant cochain map, $\psi_2\colon \mathcal{E}(2)\otimes\widetilde{N}^*_{\overline{p}}(\underline{K})\otimes\widetilde{N}^*_{\overline{q}}(\underline{K}) \to \widetilde{N}^*_{\overline{p}+\overline{q}}(\underline{K})$. Such a map is called *a structure of perverse* $\mathcal{E}(2)$*-algebra on* $\widetilde{N}^*_{\bullet}(\underline{K})$*;* its construction comes from the existence of a diagonal on $\mathcal{E}(2)$, established in [1]. Moreover, we prove in Theorem A that the cup$_i$-products arising from the existence of ψ_2 verify the two following properties, $x\cup_{|x|} x = x$ and $x\cup_i x' = 0$, if $i\geq \min(|x|,|x'|)$.

The definition of perverse $\mathcal{E}(2)$-algebras can be extended to perverse $\mathcal{E}(n)$-algebras, for any n. As this work is concerned with Steenrod squares, we consider only perverse $\mathcal{E}(2)$-algebras over $\mathbb{F}_2$. Nevertheless, it is clear that our methods of proof can be enhanced to give a structure of perverse E_∞-algebras over $\mathbb{Z}$ on $\widetilde{N}^*_{\bullet}(\underline{K})$. We will come back on these points in a forthcoming paper.

As usual, Steenrod squares are defined on $H^k_{\overline{p}}(\underline{K};\mathbb{F}_2)$ by $\mathrm{Sq}^i(x) = x\cup_{k-i} x$. Using May's presentation of Steenrod squares in [14], we see that the classical properties of Steenrod squares are a direct consequence of the structure of perverse $\mathcal{E}(2)$-algebra. We collect them, together with Adem and Cartan relations, in Theorem B. (One may observe that the proof of the Adem relation on a tensor product needs a brief incursion in the world of perverse $\mathcal{E}(4)$-algebras over $\mathbb{F}_2$.)

In Theorem B, we answer also positively to the problem asked by Goresky in [7, Page 493] and to the conjecture made by Goresky and Pardon in [10, Conjecture 7.5]. This problem concerns the range of the perversities: with the definition of Steenrod squares via the cup$_i$-products, it is clear that Sq^i sends $H^k_{\overline{p}}(\underline{K};\mathbb{F}_2)$ into $H^{k+i}_{2\overline{p}}(\underline{K};\mathbb{F}_2)$. We prove that, in fact, there is a lifting as a map $\mathrm{Sq}^i\colon H^k_{\overline{p}}(\underline{K};\mathbb{F}_2) \to H^{k+i}_{\mathcal{L}(\overline{p},i)}(\underline{K};\mathbb{F}_2)$, where $\mathcal{L}(\overline{p},i)$ is a loose perversity defined by $\mathcal{L}(\overline{p},i)(\ell) = \min(2\overline{p}(\ell),\overline{p}(\ell)+i)$, which is exactly [10, Conjecture 7.5]. This reveals an important fact because it allows the lifting of Wu classes in intersection cohomology, in a lower part of the poset of perversities.

Finally, in Theorem C, we prove that if $\underline{K}$ is the singular filtered face set associated to a pseudomanifold, X, our definition of Steenrod squares coincide with Goresky's definition introduced in [7]. For this proof, we transform the blow-up, $\widetilde{N}^*$, in a sheaf $\mathbf{IN}^*$ on X and proves that $\mathbf{IN}^*$ is isomorphic to the Deligne sheaf, in the derived category of sheaves on X. The rest of the proof comes from a unicity theorem for Steenrod squares defined on an injective sheaf, established by M. Goresky, [7].

We end with examples of concrete computation of perverse Steenrod squares, beginning with the case of isolated singularities. From it, we are able to write the Steenrod squares on the intersection cohomology of the Thom space associated to a vector bundle, in function of the Steenrod squares of the basis and the Stiefel-Whitney classes. We detail also an example of a non trivial square, $\mathrm{Sq}^2\colon H_{\overline{p}} \to H_{\mathcal{L}(\overline{p},2)}$, whose information is lost if we consider it as values in $H_{2\overline{p}}$, showing the interest of the Goresky and Pardon's conjecture. This last example can also be seen as a tubular neighborhood of a stratum,

which is the first step in the study of multiplicative structure of intersection cohomology of pseudomanifolds.

Contents

In Section 1, we recall basic notions concerning filtered face sets and their intersection cohomology. Section 2 is devoted to the construction of a structure of perverse $\mathcal{E}(2)$-algebra on the blow-up, $\widetilde{N}^*(\underline{K})$, which corresponds to the building of cup$_i$-products. In Section 3, we establish the main properties of perverse Steenrod squares, including the proof of the perverse range conjecture due to M. Goresky and W. Pardon. The comparison between our definition and Goresky's definition of Steenrod squares, in the case of a pseudomanifold, is done in Section 4. The particular case of isolated singularities and the treatment of Steenrod squares in the intersection homology of a Thom space are presented in Section 5. Finally, Section 6 is devoted to an example of a Sq^2 in the intersection cohomology of the total space of a fibration whose fiber is a cone.

1. Blow-up and perversity

In this section, we recall the basics of a simplicial version of intersection cohomology, introduced in [4].

We fix an integer n which corresponds to the formal dimension of filtered sets and consider the category $\mathbf{\Delta}^{[n]}_{\mathcal{F}}$ whose

- objects are the join $\Delta = \Delta^{j_0} * \Delta^{j_1} * \cdots * \Delta^{j_n}$, where Δ^{j_i} is the simplex of dimension j_i, possibly empty, with the conventions $\Delta^{-1} = \emptyset$ and $\emptyset * X = X$,
- maps are the $\sigma\colon \Delta = \Delta^{j_0} * \Delta^{j_1} * \cdots * \Delta^{j_n} \to \Delta' = \Delta^{k_0} * \Delta^{k_1} * \cdots * \Delta^{k_n}$, of the shape $\sigma = *_{i=0}^{n} \partial_i$, with $\partial_i \colon \Delta^{j_i} \to \Delta^{k_i}$ an injective order-preserving map for each i.

The category $\mathbf{\Delta}^{[n],+}_{\mathcal{F}}$ is the full subcategory of $\mathbf{\Delta}^{[n]}_{\mathcal{F}}$ whose objects are the joins $\Delta^{j_0} * \Delta^{j_1} * \cdots * \Delta^{j_n}$ with $\Delta^{j_n} \neq \emptyset$, i.e., $j_n \geq 0$. To any such element, we associate its *blow-up* which is the map

$$\mu\colon \widetilde{\Delta} = c\Delta^{j_0} \times \cdots \times c\Delta^{j_{n-1}} \times \Delta^{j_n} \to \Delta = \Delta^{j_0} * \cdots * \Delta^{j_n},$$

defined by

$$\begin{aligned}\mu([y_0,s_0],\dots,[y_{n-1},s_{n-1}],y_n) &= s_0y_0+(1-s_0)s_1y_1+\cdots\\ &\quad +(1-s_0)\cdots(1-s_{n-2})s_{n-1}y_{n-1}\\ &\quad +(1-s_0)\cdots(1-s_{n-2})(1-s_{n-1})y_n,\end{aligned}$$

where $y_i \in \Delta^{j_i}$ and $[y_i,s_i]\in c\Delta^{j_i}$. The prism $\widetilde{\Delta}$ is sometimes also called the blow-up of Δ.

Definition 1.1. A *filtered face set* is a contravariant functor, $\underline{K}$, from the category $\mathbf{\Delta}^{[n]}_{\mathcal{F}}$ to the category of sets, i.e., $(j_0,\dots,j_n)\mapsto \underline{K}_{(j_0,\dots,j_n)}$. The restriction of the filtered face set, $\underline{K}$, to $\mathbf{\Delta}^{[n],+}_{\mathcal{F}}$ is denoted $\underline{K}_+$.

If $\underline{K}$ and $\underline{K}'$ are filtered face sets, a *filtered face map,* $\underline{f}\colon \underline{K}\to\underline{K}'$, is a natural transformation between the two functors $\underline{K}$ and $\underline{K}'$. We denote by $\mathbf{\Delta}^{[n]}_{\mathcal{F}}-\mathit{Sets}$ the category of filtered face maps.

Remark 1.2. For any pseudomanifold, X, we can define (see [4, Example 4.5]) the singular filtered face set by

$$\underline{\mathrm{ISing}}^{\mathcal{F}}(X)_{j_0,\dots,j_n}=\{\sigma\colon \Delta^{j_0}*\cdots*\Delta^{j_n}\to X\mid \sigma^{-1}X_i=\Delta^{j_0}*\cdots*\Delta^{j_i}\},$$

for any $(j_0,\dots,j_n)$.

To any simplicial set, Y, we can associate the free $\mathbb{F}_2$-vector space $C_d(Y)$ generated by the d-dimensional simplices of Y. The normalized chain complex, $N_d(Y)$, is the quotient of $C_d(Y)$ by the degeneracies $\mathfrak{s}_i$,

$$N_d(Y)=C_d(Y)/\mathfrak{s}_0C_{d-1}(Y)+\cdots+\mathfrak{s}_{d-1}C_{d-1}(Y).$$

We consider also the dual $N^*(Y)=\hom_{\mathbb{F}_2}(N_*(Y),\mathbb{F}_2)$.

To define *the blow-up,* $\widetilde{N}^*(\underline{K})$, of N^* over a filtered face set $\underline{K}$, we first associate to any simplex $\sigma\colon\Delta^{j_0}*\cdots*\Delta^{j_n}\to\underline{K}_+$ the complex

$$\widetilde{N}^*_\sigma=N^*(c\Delta^{j_0})\otimes\cdots\otimes N^*(c\Delta^{j_{n-1}})\otimes N^*(\Delta^{j_n}).$$

A global section (or *cochain*) on $\underline{K}$, $c\in\widetilde{N}^*(\underline{K})$, is a function which assigns to each simplex $\sigma\in\underline{K}_+$ an element $c_\sigma\in\widetilde{N}^*_\sigma$ such that $c_{\partial_i\sigma}=\partial_i(c_\sigma)$ for all σ and all maps of $\mathbf{\Delta}^{[n],+}_{\mathcal{F}}$. Global sections have an extra degree, called the *perverse degree*, that we describe now.

Let $\sigma\colon\Delta^{j_0}*\cdots*\Delta^{j_n}\to\underline{K}_+$ and $\ell\in\{1,\dots,n\}$ such that $\Delta^{j_{n-\ell}}\neq\emptyset$. For any cochain $c_\sigma\in N^*(c\Delta^{j_0})\otimes\cdots\otimes N^*(c\Delta^{j_{n-1}})\otimes N^*(\Delta^{j_n})$, its restriction

$$c_{\sigma,n-\ell}\in N^*(c\Delta^{j_0})\otimes\cdots\otimes N^*(\Delta^{j_{n-\ell}}\times\{1\})\otimes\cdots\otimes N^*(c\Delta^{j_{n-1}})\otimes N^*(\Delta^{j_n}) \tag{1}$$

can be written $c_{\sigma,n-\ell}=\sum_k c'_{\sigma,n-\ell}(k)\otimes c''_{\sigma,n-\ell}(k)$, with

- $c'_{\sigma,n-\ell}(k)\in N^*(c\Delta^{j_0})\otimes\cdots\otimes N^*(c\Delta^{j_{n-\ell-1}})\otimes N^*(\Delta^{j_{n-\ell}}\times\{1\})$ and
- $c''_{\sigma,n-\ell}(k)\in N^*(c\Delta^{j_{n-\ell+1}})\otimes\cdots\otimes N^*(\Delta^{j_n})$.

Observe that each term of the tensor product in Formula (1) has a finite canonical basis and the decomposition of $c_{\sigma,n-\ell}$ can be canonically chosen in function of the associated basis of the tensor product.

Definition 1.3. If $c_{\sigma,n-\ell}\neq 0$, the *ℓ-perverse degree,* $\|c_\sigma\|_\ell$, of c_σ is equal to
$$\|c_\sigma\|_\ell=\sup_k\left\{|c''_{\sigma,n-\ell}(k)| \text{ such that } c'_{\sigma,n-\ell}(k)\neq 0\right\}.$$
If $c_{\sigma,n-\ell}=0$ or $\Delta^{j_{n-\ell}}=\emptyset$, we set $\|c_\sigma\|_\ell=-\infty$. The *perverse degree* of a global section $c\in\widetilde{N}^*(\underline{K})$ is the n-uple
$$\|c\|=(\|c\|_1,\ldots,\|c\|_n),$$
where $\|c\|_\ell$ is the supremum of the $\|c_\sigma\|_\ell$ for all $\sigma\in\underline{K}_+$.

Intersection cohomology requires a notion of perversity that we introduce now, following the convention of [12].

Definition 1.4. A *loose perversity* is a map $\overline{p}\colon\mathbb{N}\to\mathbb{Z}$, $i\mapsto\overline{p}(i)$, such that $\overline{p}(0)=0$. A *perversity* is a loose perversity such that $\overline{p}(i)\leq\overline{p}(i+1)\leq\overline{p}(i)+1$, for all $i\in\mathbb{N}$. A *Goresky-MacPherson perversity* (or *GM-perversity*) is a perversity such that $\overline{p}(1)=\overline{p}(2)=0$.

If $\overline{p}_1$ and $\overline{p}_2$ are two loose perversities, we set $\overline{p}_1\leq\overline{p}_2$ if we have $\overline{p}_1(i)\leq\overline{p}_2(i)$, for all $i\in\mathbb{N}$. The poset of all loose perversities is denoted $\mathcal{P}^n_{\text{loose}}$.

The lattice of GM-perversities, denoted $\mathcal{P}^n$, admits a maximal element, $\overline{t}$, called the *top perversity* and defined by $\overline{t}(i)=i-2$, if $i\geq 2$, $\overline{t}(0)=\overline{t}(1)=0$.

To these perversities, we add an element, $\overline{\infty}$, which is the constant map on ∞. We call it the *infinite perversity* despite the fact that it is not a perversity in the sense of the previous definition. Finally, we set $\hat{\mathcal{P}}^n=\mathcal{P}^n\cup\{\overline{\infty}\}$.

Definition 1.5. Let $\overline{p}$ be a loose perversity. A global section $c\in\widetilde{N}(\underline{K})$ is *$\overline{p}$-admissible* if $\|c\|_i\leq\overline{p}(i)$, for any $i\in\{1,\ldots,n\}$. A global section c is of *$\overline{p}$-intersection* if c and dc are $\overline{p}$-admissible.

We denote by $\widetilde{N}_{\overline{p}}(\underline{K})$ the complex of global sections of $\overline{p}$-intersection and by $H^*_{\overline{p}}(\underline{K};R)$ its homology, called the *intersection cohomology of $\underline{K}$ with coefficients in R, for the loose perversity $\overline{p}$.*

When $\overline{p}$ is a perversity and $\underline{K}=\underline{\mathrm{ISing}}^{\mathcal{F}}(X)$, with X a pseudomanifold, the complex $\widetilde{N}_{\overline{p}}(\underline{K})$ gives the same cohomology than the original Goresky-McPherson intersection cohomology of X ([8]), as it is proved in [4, Theorem D], where N^* is substituted by the non-normalized cochain complex.

2. Perverse $\mathcal{E}(2)$-algebras and filtered face sets

Steenrod squares are built from an action of a normalized homogeneous Bar resolution, $\mathcal{E}(2)$, of the symmetric group Σ_2, on the normalized singular cochains. This is the way the non-commutativity is controlled up to higher coherent homotopies and this action enriches the multiplicative structure given by the cup-product. We first review it in order to adapt this construction to the perverse setting.

The existence of an $\mathcal{E}(2)$-algebra structure on the normalized complex of a simplicial set is well known (see [14] or [1] for instance). Recall that the resolution $\mathcal{E}(2)$ is defined by
$$\ldots\mathcal{E}(2)_i\xrightarrow{d}\mathcal{E}(2)_{i-1}\to\cdots$$

with $\mathcal{E}(2)_i = \mathbb{F}_2(e_i, \tau_i)$, $de_i = d\tau_i = e_{i-1} + \tau_{i-1}$. (As we are using cochain complexes, $\mathcal{E}(2)$ is negatively graded.) The complex $\mathcal{E}(2)$ is equipped with a Σ_2-equivariant diagonal, $\mathcal{D}\colon \mathcal{E}(2) \to \mathcal{E}(2) \otimes \mathcal{E}(2)$, defined by

$$\mathcal{D}(e_i) = \sum_{j=0}^{i} e_j \otimes \tau^j . e_{i-j},$$

with $\tau . e_k = \tau_k$. This diagonal is essential for the definition of the structure of $\mathcal{E}(2)$-algebra on the normalized cochains of the blow-up's $\widetilde{\Delta}$ introduced in Section 1.

Definition 2.1. An $\mathcal{E}(2)$-*algebra structure* on a cochain complex, A^*, is a cochain map, $\psi\colon \mathcal{E}(2) \otimes A^{\otimes 2} \to A$, which is Σ_2-equivariant.

If we denote $\psi(e_i \otimes x_1 \otimes x_2)$ by $x_1 \cup_i x_2$, the previous definition is equivalent to

(1) $\psi(\tau_i \otimes x_1 \otimes x_2) = x_2 \cup_i x_1$
(2) together with the Leibniz condition:
$d(x_1 \cup_i x_2) = x_1 \cup_{i-1} x_2 + x_2 \cup_{i-1} x_1 + dx_1 \cup_i x_2 + x_1 \cup_i dx_2.$

This means that an $\mathcal{E}(2)$-algebra structure is given by a cochain map, called cup$_i$-product, $\cup_i\colon A^r \otimes A^s \to A^{r+s-i}$, satisfying the previous Leibniz condition.

Recall ([4]) that a *perverse cochain complex* is a functor from $\hat{\mathcal{P}}^n$ with values in the cochain complexes. A functor from $\mathcal{P}^n_{\text{loose}}$ with values in the cochain complexes is called a *generalized perverse cochain complex.*

Definition 2.2. A *perverse* $\mathcal{E}(2)$-*algebra structure* on a generalized perverse cochain complex, $A^*_\bullet$, is a cochain map, $\psi\colon \mathcal{E}(2) \otimes A^*_{\overline{p}} \otimes A^*_{\overline{q}} \to A^*_{\overline{p}+\overline{q}}$, which is Σ_2-equivariant.

Equivalently, a *perverse* $\mathcal{E}(2)$-*algebra structure* on $A^*_\bullet$ is entirely determined by maps, called perverse cup$_i$-products, $\cup_i\colon A^r_{\overline{p}} \otimes A^s_{\overline{q}} \to A^{r+s-i}_{\overline{p}+\overline{q}}$, satisfying the previous Leibniz condition.

As it is established by May in [14], classical properties of cup$_i$-products are a direct consequence of this $\mathcal{E}(2)$-algebra structure, except two of them that we quote in the next definition.

Definition 2.3. An $\mathcal{E}(2)$-algebra, A^*, is *nice* if it verifies the two next properties, for all x, $x' \in A$ of respective degrees $|x|$ and $|x'|$,

(i) $x \cup_{|x|} x = x$,
(ii) $x \cup_i x' = 0$ if $i > \min(|x|, |x'|)$.

Nice perverse $\mathcal{E}(2)$-*algebras* are defined similarly.

Observe the useful next property of nice $\mathcal{E}(2)$-algebras.

Lemma 2.4. *Let A be a nice $\mathcal{E}(2)$-algebra. If $a \in A^d$ and $b \in A^d$, we have*

$$a \cup_d b = b \cup_d a.$$

Proof. Property (ii) of Definition 2.3 and Leibniz rule imply

$$\begin{aligned} d(a \cup_{d+1} b) &= 0 \\ &= a \cup_d b + b \cup_d a + da \cup_{d+1} b + a \cup_{d+1} db \\ &= a \cup_d b + b \cup_d a. \end{aligned}$$

$\square$

We prove now that the blow-up of normalized cochain of filtered face sets fulfils these conditions. Let $\Delta = \Delta^{j_0} * \Delta^{j_1} * \cdots * \Delta^{j_n}$ and let $\mathcal{D}$ be the diagonal of $\mathcal{E}(2)$. The coassociativity of $\mathcal{D}$ allows an easy iteration, and we have

$$\mathcal{D}^{n-1}(e_i) = \sum_{(i_1,\dots,i_n) \text{ with } i_1+\cdots+i_n=i} e_{i_1} \otimes \tau^{i_1}.e_{i_2} \otimes \cdots \otimes \tau^{i_1+\cdots+i_{n-1}}.e_{i_n}.$$

With this diagonal, the action of $\mathcal{E}(2)$ on the normalized cochain complex of a simplicial set can be extended in an action of $\mathcal{E}(2)$ on $\widetilde{N}^*(\Delta)$ with the map

$$\begin{array}{c} \mathcal{E}(2) \otimes N^*(c\Delta^{j_1})^{\otimes 2} \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}})^{\otimes 2} \otimes N^*(\Delta^{j_n})^{\otimes 2} \\ \Big\downarrow \Phi \\ N^*(c\Delta^{j_1}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n}), \end{array}$$

which sends the element $e_i \otimes (x_1 \otimes y_1) \otimes \cdots \otimes (x_n \otimes y_n)$ in the domain of Φ to

$$\sum_{(i_1,\dots,i_n) \text{ with } i_1+\cdots+i_n=i} (x_1 \cup_{i_1} y_1) \otimes \tau^{i_2}.(x_2 \cup_{i_2} y_2) \otimes \cdots \otimes \tau^{i_1+\cdots+i_{n-1}}.(x_n \cup_{i_n} y_n),$$

where $\tau.(x \cup_i y) = y \cup_i x$.

Theorem A. *Let $\underline{K}$ be a filtered face set and $\overline{p}$ be a loose perversity. The generalized perverse cochain complex, $\overline{p} \mapsto \widetilde{N}^*(\underline{K})_{\overline{p}}$, is a nice perverse $\mathcal{E}(2)$-algebra.*

Lemma 2.5. *Any tensor product of nice $\mathcal{E}(2)$-algebras is a nice $\mathcal{E}(2)$-algebra for the product structure coming from the diagonal of $\mathcal{E}(2)$.*

Proof. By coassociativity of the diagonal of $\mathcal{E}(2)$, it is sufficient to reduce the proof to the case of the tensor product of two nice $\mathcal{E}(2)$-algebras, A and B.

Let $x = \sum_k a_k \otimes b_k \in (A \otimes B)^d$ and $x' = \sum_\ell a'_\ell \otimes b'_\ell \in (A \otimes B)^{d'}$ with $d \leq d'$. We set $f = d + m$ with $m \geq 0$. One compute

$$x \cup_f x' = \sum_{f_1+f_2=f} \sum_{k,\ell} (a_k \cup_{f_1} a'_\ell) \otimes \tau^{f_1}.(b_k \cup_{f_2} b'_\ell).$$

If the element $(a_k \cup_{f_1} a'_\ell) \otimes \tau^{f_1}.(b_k \cup_{f_2} b'_\ell)$ of this sum is not equal to zero, we must have

$$f_1 \leq \min(|a_k|, |a'_\ell|) \text{ and } f_2 \leq \min(|b_k|, |b'_\ell|),$$

which implies $f = f_1 + f_2 = d + m \leq |a_k| + |b_k| = d$ and $m = 0$. We have established Property (ii) of Definition 2.3.

As for Property (i), we consider

$$x \cup_d x = \sum_{f_1+f_2=d} \sum_{k,k'} (a_k \cup_{f_1} a_{k'}) \otimes \tau^{f_1}.(b_k \cup_{f_2} b_{k'}).$$

As above, if the element $(a_k \cup_{f_1} a_{k'}) \otimes \tau^{f_1}.(b_k \cup_{f_2} b_{k'})$ of this sum is not equal to zero, we must have

$$f_1 \leq \min(|a_k|, |a_{k'}|) \text{ and } f_2 \leq \min(|b_k|, |b_{k'}|).$$

Suppose $\min(|a_k|, |a_{k'}|) = |a_k|$, then we have $|b_{k'}| \leq |b_k|$, because $|a_k|+|b_k| = |a_{k'}|+|b_{k'}|$, and also $d = |a_k| + |b_k| = f_1 + f_2 \leq |a_k| + |b_{k'}|$, which imply $|b_k| = |b_{k'}|$. Therefore, the non-zero elements of this sum must be of the shape $(a_k \cup_{d-r} a_{k'}) \otimes \tau^{f_1}.(b_k \cup_r b_{k'})$ with $|a_k| = |a_{k'}| = d - r$, $|b_k| = |b_{k'}| = r$.

With Lemma 2.4, if $a_k \neq a_{k'}$, the same term appears twice, as $(a_k \cup_{d-r} a_{k'}) \otimes (b_k \cup_r b_{k'})$ and as $(a_{k'} \cup_{d-r} a_k) \otimes (b_k \cup_r b_{k'})$. Their sum is equal to zero. With the same argument applied to the case $b_k \neq b_{k'}$, we have reduced the previous expression to

$$\begin{aligned} x \cup_d x &= \sum_k (a_k \cup_{d-r} a_k) \otimes (b_k \cup_r b_k) \\ &= \sum_k a_k \otimes b_k = x, \end{aligned}$$

and Property (i) of Definition 2.3 is established. □

Proof of Theorem A. A cochain $c \in \widetilde{N}^*(\underline{K})$ associates to any simplex, $\sigma\colon \Delta = \Delta^{j_0} * \cdots * \Delta^{j_n} \to \underline{K}_+$, an element $c_\sigma \in N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$.

If we set $(c \cup_i c')_\sigma = c_\sigma \cup_i c'_\sigma$, by naturality of the structure of $\mathcal{E}(2)$-algebra on $N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$, we get a global section $c \cup_i c' \in \widetilde{N}^*(\underline{K})$. More precisely, we have a Σ_2-equivariant cochain map,

$$\mathcal{E}(2) \otimes \widetilde{N}^*(\underline{K})^{\otimes 2} \to \widetilde{N}^*(\underline{K}),$$

entirely defined by $e_i \otimes c \otimes c' \mapsto c \cup_i c'$, which gives to $\widetilde{N}^*(\underline{K})$ a structure of $\mathcal{E}(2)$-algebra. The niceness of this structure is a direct consequence of Lemma 2.5.

We study now the behavior of this structure with the perverse degree. The perversity degree being a local notion, we consider c and c' in $N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$, with $j_n \geq 0$, and $\ell \in \{1, \ldots, n\}$ such that $\Delta^{j_{n-\ell}} \neq \emptyset$. We denote by $c_{n-\ell}$ and $c'_{n-\ell}$ the respective restrictions of c and c' to $N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(\Delta^{j_{n-\ell}} \times \{1\}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$.

We decompose c, c' in $c = \sum_{s=0}^m c_1^s \otimes \cdots \otimes c_n^s$, $c' = \sum_{t=0}^{m'} c_1'^t \otimes \cdots \otimes c_n'^t$ and their restriction in $c_{n-\ell} = \sum_{s=0}^m c_1^s \otimes \cdots \otimes \Phi^*_{n-\ell} c^s_{n-\ell} \otimes \cdots \otimes c_n^s$, $c'_{n-\ell} = \sum_{t=0}^m c_1'^t \otimes \cdots \otimes \Phi^*_{n-\ell} c'^t_{n-\ell} \otimes \cdots \otimes c_n'^t$, where $\Phi^*_{n-\ell}$ is induced by the inclusion $\Delta^{j_{n-\ell}} \times \{1\} \hookrightarrow c\Delta^{j_{n-\ell}}$. By definition, we have

$$\|c\|_\ell = \sup_s \{|c^s_{n-\ell+1} \otimes \cdots \otimes c^s_n| \text{ such that } c_1^s \otimes \cdots \otimes \Phi^*_{n-\ell} c^s_{n-\ell} \neq 0\}.$$

Let

$$\begin{array}{c} N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-\ell}}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n}) \\ \Big\downarrow {\scriptstyle \hat{\Phi}^*_{n-\ell} = \mathrm{id} \otimes \Phi^*_{n-\ell} \otimes \mathrm{id}} \\ N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(\Delta^{j_{n-\ell}} \times \{1\}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n}). \end{array}$$

As the cup$_i$-product is natural, we have $\hat{\Phi}^*_{n-\ell}(c \cup_i c') = \hat{\Phi}^*_{n-\ell}(c) \cup_i \hat{\Phi}^*_{n-\ell}(c')$.

- If $\hat{\Phi}^*_{n-\ell}(c) = 0$ or $\hat{\Phi}^*_{n-\ell}(c') = 0$, we have $\hat{\Phi}^*_{n-\ell}(c) \cup_i \hat{\Phi}^*_{n-\ell}(c') = 0$ and thus

$$\|c \cup_i c'\|_\ell = -\infty.$$

- Suppose now $\hat{\Phi}^*_{n-\ell}(c) \neq 0$ and $\hat{\Phi}^*_{n-\ell}(c') \neq 0$. By definition of the cup$_i$-product, $\hat{\Phi}^*_{n-\ell}(c) \cup_i \hat{\Phi}^*_{n-\ell}(c')$ is a sum of tensor products whose elements are of two kinds:
  (1) $c^s_j \cup_{f_j} c'^t_j$, with $j \neq n-\ell$, or
  (2) $\Phi^*_{n-\ell}(c^s_{n-\ell}) \cup_{f_{n-\ell}} \hat{\Phi}^*_{n-\ell}(c'^t_{n-\ell})$.

As $|c^s_j \cup_{f_j} c'^t_j| = |c^s_j| + |c'^t_j| - f_j$, the cochain degree decreases and we obtain, for each ℓ,

$$\|c \cup_i c'\|_\ell \leq \|c\|_\ell + \|c'\|_\ell,$$

by definition of the perverse degree, see Definition 1.3. Therefore, we have

$$\|c \cup_i c'\| \leq \|c\| + \|c'\|.$$

Now, the rule of Leibniz implies

$$\|d(c \cup_i c')\| \leq \max(\|dc\| + \|c'\|, \|dc'\| + \|c\|, \|c\| + \|c'\|).$$

Thus, if $\|c\| \leq \overline{p}$, $\|dc\| \leq \overline{p}$, $\|c'\| \leq \overline{q}$ and $\|dc'\| \leq \overline{q}$, we have $\|c \cup_i c'\| \leq \overline{p} + \overline{q}$ and $\|d(c \cup_i c')\| \leq \overline{p} + \overline{q}$. This implies that the $\mathcal{E}(2)$-algebra structure on $\widetilde{N}^*(\underline{K})$ induces equivariant cochain maps

$$\mathcal{E}(2) \otimes \widetilde{N}^*_{\overline{p}}(\underline{K}) \otimes \widetilde{N}^*_{\overline{q}}(\underline{K}) \to \widetilde{N}^*_{\overline{p}+\overline{q}}(\underline{K}).$$

That means: $\widetilde{N}^*_\bullet$ is a perverse $\mathcal{E}(2)$-algebra. □

3. Steenrod perverse squares

From the existence of perverse cup$_i$-products, we define Steenrod squares, as in the classical case. In the next statement, when $i > 0$, the fact that the loose perversity image of Sq^i is $\mathcal{L}(\overline{p}, i)$, defined by $\mathcal{L}(\overline{p}, i)(\ell) = \min(2\overline{p}(\ell), \overline{p}(\ell) + i)$, answers positively a conjecture of Goresky and Pardon, see [10, Conjecture 7.5]. More explicitly, we prove the existence of a dashed arrow which lifts the square Sq^i,

$$\begin{tikzcd}
& H^r_{\mathcal{L}(\overline{p},i)} \arrow[d] \\
H^r_{\overline{p}} \arrow[ur, dashed] \arrow[r, "\mathrm{Sq}^i"] & H^r_{2\overline{p}}
\end{tikzcd}$$

Theorem B. *Let $\underline{K}$ be a filtered face set and $\overline{p}$ be a loose perversity. The perverse cup$_i$-products induce perverse squares, defined by $\mathrm{Sq}^i(x) = x \cup_{|x|-i} x$, for $x \in H^{|x|}_{\overline{p}}(\underline{K}; \mathbb{F}_2)$, which satisfy the following properties.*

(1) If $i < 0$, then $\mathrm{Sq}^i(x) = 0$.

(2) If $i \geq 0$, then we have

$$\mathrm{Sq}^i \colon H^r_{\overline{p}}(\underline{K}; \mathbb{F}_2) \to H^{r+i}_{\mathcal{L}(\overline{p},i)}(\underline{K}; \mathbb{F}_2),$$

where $\mathcal{L}(\overline{p}, i) = \min(2\overline{p}, \overline{p} + i)$ and
  (i) $\mathrm{Sq}^i(x) = 0$ *if* $i > |x|$,
  (ii) $\mathrm{Sq}^{|x|}(x) = x^2$,
  (iii) $\mathrm{Sq}^0 = \mathrm{id}$.

(iv) If $x \in H^{|x|}_{\overline{p}}(\underline{K};\mathbb{F}_2)$, $y \in H^{|y|}_{\overline{q}}(\underline{K};\mathbb{F}_2)$, one has the Cartan formula,

$$\mathrm{Sq}^i(x\cup y) = \sum_{i_1+i_2=i} \mathrm{Sq}^{i_1}(x)\cup \mathrm{Sq}^{i_2}(y) \in H^{|x|+|y|+i}_{\overline{r}}(\underline{K};\mathbb{F}_2),$$

with $\overline{r} = \min(2\overline{p}+2\overline{q}, \overline{p}+\overline{q}+i)$.

(v) For any pair (i,j), with $i<2j$, one has the Adem relation:

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$$

and $\mathrm{Sq}^i\mathrm{Sq}^j$ sends $H^*_{\overline{p}}$ in $H^{*+i+j}_{\overline{r}}$, with $\overline{r} = \min(4\overline{p}, 2\overline{p}+i, \overline{p}+i+j)$.

Before proving this theorem, we establish a technical property on the tensor product of two nice $\mathcal{E}(2)$-algebras.

Lemma 3.1. *Let A and B be two nice $\mathcal{E}(2)$-algebras and $A\otimes B$ their tensor product equipped with the $\mathcal{E}(2)$-algebra structure coming from the diagonal of $\mathcal{E}(2)$. Let x, x' in A, y, y' in B such that $|x|+|y| = |x'|+|y'| = d$, $|y|\leq r$ and $|y'|\leq r$. Then, for any $k\in\{0,\ldots,d-i\}$ such that $x\cup_{d-k-i} x'\otimes y\cup_k y'\neq 0$, we have $|y\cup_k y'|\leq r+i$.*

Proof. If $x\cup_{d-k-i} x'\otimes y\cup_k y'\neq 0$, we must have $k\leq \min(|y|,|y'|)$ and $d-k-i\leq \min(|x|,|x'|)$, which implies

$$d-i-\min(|x|,|x'|)\leq k.$$

Suppose $\min(|x|,|x'|)=|x|$. Then we have

$$\begin{aligned} |y|+|y'|-d+i+\min(|x|,|x'|) &= |y|+|y'|-(|x|+|y|)+i+|x| \\ &= |y'|+i, \end{aligned}$$

which implies

$$|y\cup_k y'| = |y|+|y'|-k \leq |y'|+i\leq r+i.$$

□

Directly from the definition of cup$_k$-product, the inequalities $|y|\leq r$ and $|y'|\leq r$ imply $|y\cup_k y'|\leq 2r$. Thus, the majoration $|y\cup_k y'|\leq r+i$ obtained in Lemma 3.1 is exactly what is needed for the proof of the conjecture of Goresky and Pardon, as we show in the beginning of the next proof.

Proof of Theorem B. Let $i\geq 0$. From their definition as particular cup$_i$-products, the Steenrod squares have their image in the intersection cohomology with loose perversity $2\overline{p}$. We prove first that the loose perversity $2\overline{p}$ can be replaced by $\mathcal{L}(\overline{p},i)$. We take over the arguments and the method used at the end of the proof of Theorem A by considering a cocycle $c\in N^*(c\Delta^{j_0})\otimes\cdots\otimes N^*(c\Delta^{j_{n-1}})\otimes N^*(\Delta^{j_n})$, $\ell\in\{1,\ldots,n\}$, such that $\Delta^{j_{n-\ell}}\neq\emptyset$, and the restriction $c_{n-\ell}$ of c to $N^*(c\Delta^{j_0})\otimes\cdots\otimes N^*(\Delta^{j_{n-\ell}}\times\{1\})\otimes\cdots\otimes N^*(c\Delta^{j_{n-1}})\otimes N^*(\Delta^{j_n})$. Observe first that, by naturality, we have $(c\cup_{|c|-i} c)_{n-\ell} = c_{n-\ell}\cup_{|c_{n-\ell}|-i} c_{n-\ell}$.

• If $c_{n-\ell}=0$, we have $(c\cup_{|c|-i} c)_{n-\ell}=0$ and $\|c_{n-\ell}\cup_{|c_{n-\ell}|} c_{n-\ell}\| = \|c\cup_{|c|-i} c\| = -\infty$.

• If $c_{n-\ell}\neq 0$, we decompose it in a canonical form, $c_{n-\ell} = \sum_s c'^s_{n-\ell}\otimes c''^s_{n-\ell}\in A\otimes B$, with $A = N^*(c\Delta^{j_0})\otimes\cdots\otimes N^*(\Delta^{j_{n-\ell}}\times\{1\})$ and $B = N^*(c\Delta^{j_{n-\ell+1}})\otimes\cdots\otimes N^*(c\Delta^{j_{n-1}})\otimes N^*(\Delta^{j_n})$. Using Lemma 3.1, we know that $(c'^s_{n-\ell}\cup_{|c_{n-\ell}|-k-i} c'^t_{n-\ell})\otimes(c''^s_{n-\ell}\cup_k c''^t_{n-\ell})\neq 0$

implies $|c''^s_{n-\ell} \cup_k c''^t_{n-\ell}| \leq \overline{p}(\ell) + i$, for any pair of indices, (s,t), in the writing of $c_{n-\ell}$. This infers $\|c \cup_{|c|-i} c\| \leq \overline{p} + i$, as announced.

The condition on the perversity of the differential of $c \cup_{|c|-i} c$ is immediate here because c is a cocycle.

The list *(1)*, *(2)-(i)*, *(2)-(ii)*, *(2)-(iii)* of properties is a direct consequence of Theorem A and [14, Section 5].

Let A and B be two nice $\mathcal{E}(2)$-algebras. By definition of the diagonal action of $\mathcal{E}(2)$ on the tensor product, we have a Cartan external formula

$$\mathrm{Sq}^i(a \otimes b) = \sum_{i_1+i_2=i} \mathrm{Sq}^{i_1}(a) \otimes \mathrm{Sq}^{i_2}(b),$$

for $a \in A$ and $b \in B$. In our case, each factor, A and B, satisfies the Cartan internal formula. Therefore, the Cartan internal formula on $A \otimes B$ is a direct consequence of the next equalities:

$$\begin{aligned}
\mathrm{Sq}^i((a \otimes b) \cup (a' \otimes b')) &=_{(1)} \mathrm{Sq}^i((a \cup a') \otimes (b \cup b')) \\
&=_{(2)} \sum_{i_1+i_2=i} \mathrm{Sq}^{i_1}(a \cup a') \otimes \mathrm{Sq}^{i_2}(b \cup b') \\
&=_{(3)} \sum_{j_1+j_2+k_1+k_2=i} (\mathrm{Sq}^{j_1}(a) \cup \mathrm{Sq}^{j_2}(a')) \otimes (\mathrm{Sq}^{k_1}(b) \cup \mathrm{Sq}^{k_2}(b'))
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i_1+i_2=i} \mathrm{Sq}^{i_1}(a \otimes b) \cup \mathrm{Sq}^{i_2}(a' \otimes b') &=_{(2)} \sum_{s_1+s_2+t_1+t_2=i} (\mathrm{Sq}^{s_1}(a) \otimes \mathrm{Sq}^{s_2}(b)) \cup \mathrm{Sq}^{t_1}(a') \otimes \mathrm{Sq}^{t_2}(b')) \\
&=_{(1)} \sum_{s_1+s_2+t_1+t_2=i} (\mathrm{Sq}^{s_1}(a) \cup \mathrm{Sq}^{t_1}(a')) \otimes (\mathrm{Sq}^{s_2}(b) \cup \mathrm{Sq}^{t_2}(b')),
\end{aligned}$$

where $=_{(1)}$ comes from the definition of the cup-product on a tensor product, $=_{(2)}$ from the application of the Cartan external formula and $=_{(3)}$ from the Cartan internal formula on each factor.

For the Adem's formula *(2)-(v)*, we need some recalls in order to track the perversity conditions. The classical proof uses the Bar resolution, $\mathcal{E}(4)$, and the existence of a Σ_4-equivariant cochain map, $\mathcal{E}(4) \otimes N^*(K)^{\otimes 4} \to N^*(K)$, for any simplicial set K, called an $\mathcal{E}(4)$-algebra. As these objects appear just in this part of proof, we de not recall them in detail, referring to [1, Section 1]. We mention only the points related to the control of perversities.

Denote by $\omega \colon \mathcal{E}(2) \otimes \mathcal{E}(2) \otimes \mathcal{E}(2) \to \mathcal{E}(4)$ the cochain map induced by the wreath product $\Sigma_2 \times \Sigma_2 \times \Sigma_2 \to \Sigma_4$. Let A be an $\mathcal{E}(2)$ and an $\mathcal{E}(4)$-algebra whose structure maps are respectively denoted ψ_2 and ψ_4. We say that A is an Adem-object ([14]) if there is a commutative diagram

$$\begin{tikzcd}
\mathcal{E}(2) \otimes \mathcal{E}(2)^{\otimes 2} \otimes A^{\otimes 4} \arrow[r, "\omega\otimes\mathrm{id}"] \arrow[d, "\mathrm{Sh}"'] & \mathcal{E}(4) \otimes A^{\otimes 4} \arrow[r, "\psi_4"] & A \\
\mathcal{E}(2) \otimes (\mathcal{E}(2) \otimes A^{\otimes 2})^{\otimes 2} \arrow[r, "\mathrm{id}\otimes\psi_2^{\otimes 2}"'] & \mathcal{E}(2) \otimes A^{\otimes 2} \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

where Sh is the appropriate shuffle map.

Let $\Delta = \Delta^{j_0} * \cdots * \Delta^{j_n}$ and $A = N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$. Because $N^*(K)$ is an Adem-object for any simplicial set K and because the tensor product of two nice $\mathcal{E}(2)$-algebras satisfying the Adem formula is also an $\mathcal{E}(2)$-algebra satisfying the Adem formula ([14, Lemma 4.2, Page 174]), we know that A is an Adem-object. In Theorem A, we prove that ψ_2 restricts to a map $\mathcal{E}(2) \otimes A_{\overline{p}} \otimes A_{\overline{q}} \to A_{\overline{p}+\overline{q}}$. The involved argument is the fact that perversity degree is a combination of cohomological degree and restriction maps. The same reason gives a Σ_4-equivariant cochain map $\mathcal{E}(4) \otimes A_{\overline{p}_1} \otimes A_{\overline{p}_2} \otimes A_{\overline{p}_3} \otimes A_{\overline{p}_4} \to A_{\overline{p}_1+\overline{p}_2+\overline{p}_3+\overline{p}_4}$. Therefore, the Adem object A is compatible with perversities and we get an Adem formula for intersection cohomology. □

Remark 3.2. Previous definitions and results can be adapted to the context of GM-perversities. By restricting to GM-perversities $\overline{p}$ and $\overline{q}$ such that $\overline{p} + \overline{q} \leq \overline{t}$, the cup$_i$-products are defined by

$$\cup_i \colon A^r_{\overline{p}} \otimes A^s_{\overline{q}} \to A^{r+s+i}_{\overline{p}\oplus\overline{q}},$$

where the sum $\overline{p} \oplus \overline{q}$ is taken in the lattice $\mathcal{P}^n$, see [11] or [4, Section 10]. The Steenrod squares introduced in Section 3,

$$\mathrm{Sq}^i \colon H^r_{\overline{p}} \to H^{r+i}_{\overline{q}},$$

are defined for GM-perversities $\overline{p}$, $\overline{q}$ such that $\min(2\overline{p}, \overline{p} + i) \leq \overline{q}$.

4. Comparison with Goresky's construction

In [7], M. Goresky has already defined Steenrod squares, $\mathrm{Sq}^i_{\mathrm{G}}$, on the intersection cohomology, $H^*_{\mathrm{GM},\overline{p}}(X; \mathbb{Z}_2)$, of a topological pseudomanifold, in the case of a GM-perversity $\overline{p}$. In this section, we prove that the two Steenrod squares, Sq^i and $\mathrm{Sq}^i_{\mathrm{G}}$, coincide. We first recall the definition of topological pseudomanifolds.

Definition 4.1. An *n-dimensional topological pseudomanifold* is a topological space with a filtration by closed subsets,

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_{n-2} \subseteq X_n = X,$$

such that

- (i) $X_{n-k} \backslash X_{n-k-1}$ is a metrizable topological manifold of dimension $n-k$ or the empty set,
- (ii) $X_n \backslash X_{n-2}$ is dense in X,
- (iii) for each point $x \in X_i \backslash X_{i-1}$, there exist
  - (a) an open neighborhood, V, of x in X, endowed with the induced filtration,
  - (b) an open neighborhood, U, of x in $X_i \backslash X_{i-1}$,
  - (c) a compact topological pseudomanifold, L, of dimension $n - i - 1$, whose cone, $\mathring{c}L$ is endowed with the conic filtration,
  - (d) a homeomorphism, $\varphi \colon U \times \mathring{c}L \to V$, such that
    - (1) $\varphi(u, \vartheta) = u$, for any $u \in U$, with ϑ the summit of the cone,
    - (2) $\varphi(U \times L_j \times]0, 1[) = V \cap X_{i+j+1}$, for any $j \in \{0, \ldots, n - i - 1\}$.

Recall the filtered face set $\underline{\mathrm{ISing}}^{\mathcal{F}}(X)$ introduced in Remark 1.2. The next result connects Goresky's definition of Steenrod squares on the intersection cohomology of X

to our definition of Steenrod squares on the intersection cohomology of the filtered face set $\underline{\mathrm{ISing}}^{\mathcal{F}}(X)$.

Theorem C. *Let X be a topological pseudomanifold. For any GM-perversity $\overline{p}$, there exists isomorphisms, $\theta^*_{\overline{p}}\colon H^*_{\overline{p}}(\underline{\mathrm{ISing}}^{\mathcal{F}}_*(X);\mathbb{F}_2)\to H^*_{\mathrm{GM},\overline{p}}(X;\mathbb{F}_2)$, such that the following diagram commutes,*

$$\begin{tikzcd}
H^r_{\overline{p}}(\underline{\mathrm{ISing}}^{\mathcal{F}}(X);\mathbb{F}_2) \arrow[r, "\mathrm{Sq}^i"] \arrow[d, "\theta^r_{\overline{p}}"'] & H^{r+i}_{\mathcal{L}(\overline{p},i)}(\underline{\mathrm{ISing}}^{\mathcal{F}}(X);\mathbb{F}_2) \arrow[d, "\theta^{r+i}_{\mathcal{L}(\overline{p},i)}"] \\
H^r_{\mathrm{GM},\overline{p}}(X;\mathbb{F}_2) \arrow[r, "\mathrm{Sq}^i_{\mathrm{G}}"] & H^{r+i}_{\mathrm{GM},\mathcal{L}(\overline{p},i)}(X;\mathbb{F}_2).
\end{tikzcd}$$

In [7], see also [9] and [2, Chapter V], the intersection cohomology on X is introduced by the use of a sheaf due to Deligne. The Deligne's sheaf, $\mathbf{IC}_{\overline{p}}$, is defined by a sequence of truncations starting from the constant sheaf on $X_n\backslash X_{n-2}$. As we are not using this specific construction, we do not recall it, sending the reader to the previous references. From the local system $\widetilde{N}^*$, we define a presheaf on X by

$$IN^*_{\overline{p}}(U)=\widetilde{N}^*_{\overline{p}}(\underline{\mathrm{ISing}}^{\mathcal{F}}(U)),$$

for any open set U of X. Denote by $\mathrm{Cov}(U)$ the set of open covers of U and, for any $\mathcal{U}\in\mathrm{Cov}(U)$, by $\underline{\mathrm{ISing}}^{\mathcal{F},\mathcal{U}}(U)$ the sub-filtered face set of $\underline{\mathrm{ISing}}^{\mathcal{F}}(U)$ whose elements have a support included in an element of $\mathcal{U}$. The sheafification of $IN^*_{\overline{p}}$ is given by

$$\mathbf{IN}^*_{\overline{p}}(U)=\lim_{\mathcal{U}\in\mathrm{Cov}(U)}\widetilde{N}^*_{\overline{p}}(\underline{\mathrm{ISing}}^{\mathcal{F},\mathcal{U}}(U)).$$

The cup_i-products introduced in Section 3 on $\widetilde{N}^*_\bullet(\underline{\mathrm{ISing}}^{\mathcal{F},\mathcal{U}}(U))$ induce cup_i-products on $\mathbf{IN}^*_\bullet(U)$, by definition of the last one as a direct limit.

Theorem C is a direct consequence of the next two lemmas. First, we connect the definition of Steenrod squares on $\underline{\mathrm{ISing}}^{\mathcal{F}}(X)$ with a definition involving the sheaf $\mathbf{IN}^*_\bullet$ on X.

Lemma 4.2. *For any n-dimensional topological pseudomanifold, X, and any loose perversity $\overline{p}$, we have a commutative diagram,*

$$\begin{tikzcd}
H^r_{\overline{p}}(\underline{\mathrm{ISing}}^{\mathcal{F}}(X);\mathbb{F}_2) \arrow[r, "\mathrm{Sq}^i"] \arrow[d, "\cong"'] & H^{r+i}_{\overline{p}}(\underline{\mathrm{ISing}}^{\mathcal{F}}(X);\mathbb{F}_2) \arrow[d, "\cong"] \\
H^r(X;\mathbf{IN}_{\overline{p}}) \arrow[r, "\mathrm{Sq}^i"] & H^{r+i}(X;\mathbf{IN}_{\mathcal{L}(\overline{p},i)}),
\end{tikzcd}$$

in which vertical maps are quasi-isomorphisms induced by the canonical map $IN^*_\bullet\to\mathbf{IN}^*_\bullet$.

Proof. In [4, Lemma 2.13], we prove that the canonical inclusion $\underline{\mathrm{ISing}}^{\mathcal{F},\mathcal{U}}(U)\to\underline{\mathrm{ISing}}^{\mathcal{F}}(U)$ induces an isomorphism in intersection cohomology, for any loose perversity $\overline{p}$. By passage to the direct limit, we get an isomorphism

$$H^*\left(\lim_{\mathcal{U}\in\mathrm{Cov}(U)}\widetilde{N}^*_{\overline{p}}(\underline{\mathrm{ISing}}^{\mathcal{F},\mathcal{U}}(U))\right)\cong H^*\left(\widetilde{N}^*_{\overline{p}}(\underline{\mathrm{ISing}}^{\mathcal{F}}(U))\right),$$

whose righthand side is the intersection cohomology of a filtered face set.

In a second step, we prove that the sheaf, $\mathbf{IN}^*_{\overline{p}}$, is soft. For that, observe that $IN^*_{\overline{p}}(U)$ is an $IN^*_{\overline{0}}(U)$-module. On the other hand, the restriction of cochains to the vertices of the regular part, $N^0(U) \to IN^0_{\overline{0}}(U)$, is a morphism of sheaf of rings. As the sheaf N^0 is soft, and any sheaf of modules over a soft sheaf of rings is soft, we deduce the softness of $\mathbf{IN}^*_{\overline{p}}$. Thus, the hypercohomology is obtained from the sections of the sheaf and we have got a series of isomorphisms,

$$H^*(X;\mathbf{IN}^*_{\overline{p}}) \cong H^*(\Gamma(X,\mathbf{IN}^*_{\overline{p}})) \cong H^*(\widetilde{N}^*_{\overline{p}}(\underline{\mathrm{ISing}}^{\mathcal{F}}(X))) \cong H^*_{\overline{p}}(X;\mathbb{F}_2).$$

By definition of the cup$_i$-products on $\mathbf{IN}^*_\bullet$, the following diagram commutes,

$$\begin{tikzcd}
IN^r_{\overline{p}}(X) \otimes IN^s_{\overline{q}}(X) \arrow[r, "\cup_i"] \arrow[d, "\simeq"'] & IN^{r+s-i}_{\overline{p}\oplus\overline{q}}(X) \arrow[d, "\simeq"] \\
\Gamma(X,\mathbf{IN}^r_{\overline{p}}) \otimes \Gamma(X,\mathbf{IN}^s_{\overline{q}}) \arrow[r, "\cup_i"] & \Gamma(X,\mathbf{IN}^{r+s-i}_{\overline{p}\oplus\overline{q}}).
\end{tikzcd}$$

With the properties established at the beginning of this proof, the vertical maps are quasi-isomorphisms induced by the canonical map $IN^*_\bullet \to \mathbf{IN}^*_\bullet$. The stated result is now a consequence of the definition of Steenrod squares from cup$_i$-products. $\square$

The second step is the comparison of the two definitions of Steenrod squares, respectively associated to the sheaf $\mathbf{IN}^*_\bullet$ and to the Deligne sheaf $\mathbf{IC}^*_\bullet$. This is a consequence of the comparison of the two associated cup$_i$-products, done in the next lemma.

Lemma 4.3. *Let X be an n-dimensional topological pseudomanifold and let $\overline{p}$, $\overline{q}$ be two GM-perversities, such that $\overline{p}\oplus\overline{q} \leq \overline{t}$. Then, for any i, there is a commutative square in the derived category of sheaves on X, linking the two cup$_i$-products,*

$$\begin{tikzcd}
\mathbf{IN}^*_{\overline{p}}(X) \otimes \mathbf{IN}^*_{\overline{q}}(X) \arrow[r, "\cup_i"] \arrow[d, squiggly] & \mathbf{IN}^*_{\overline{p}\oplus\overline{q}}(X) \arrow[d, squiggly] \\
\mathbf{IC}^*_{\overline{p}}(X) \otimes \mathbf{IC}^*_{\overline{q}}(X) \arrow[r, "\cup_i"] & \mathbf{IC}^*_{\overline{p}\oplus\overline{q}}(X),
\end{tikzcd}$$

and such that vertical arrows are sequences of quasi-isomorphisms.

Proof. From Lemma 4.2, we know that the intersection cohomology with values in the sheaf $\mathbf{IN}^*_\bullet$ is isomorphic to the intersection cohomology of the filtered face set $\underline{\mathrm{ISing}}^{\mathcal{F}}(X)$. In [4, Theorem D], we prove, by using results of H. King ([12]), that this last cohomology is isomorphic to the M. Goresky and R. MacPherson cohomology introduced in [8]. Finally, recall that the definitions of M. Goresky and R. MacPherson given in [8] and [9] coincide (see the detailed proof of G. Friedman in [6, Section 6]).

Let $\mathbf{S}^*$ a differential graded sheaf on the pseudomanifold X. We denote by $\mathbf{S}^*_k$ the restriction of $\mathbf{S}^*$ to the open set $X\backslash X_{n-k}$, for $k \in \{2,\dots,n+1\}$. Recall the conditions (AX1) of [2, V.2.3]:

(a) $\mathbf{S}$ is bounded, $\mathbf{S}^i = 0$ for $i < 0$ and $\mathbf{S}^*_2$ is quasi-isomorphic to the ordinary singular cohomology.

(b) For any $k \in \{2,\dots,n\}$ and any $x \in X_{n-k}\backslash X_{n-k-1}$, we have $H^i(\mathbf{S}^*_x) = 0$ if $i > \overline{p}(k)$.

(c) The attachment map, $\alpha_k\colon \mathbf{S}^*_{k+1} \to \mathbb{F}_2 i_{k*}\mathbf{S}^*_k$, induced by the canonical inclusion $X\backslash X_{n-k} \to X\backslash X_{n-k-1}$, is a quasi-isomorphism up to $\overline{p}(k)$.

The sheaf $\mathbf{IN}^*_\bullet$ satisfies condition (a) by definition. Conditions (b) and (c) are cohomological properties and, from the beginning of this proof, we know the existence of an isomorphism between the intersection cohomology with values in the sheaf $\mathbf{IN}^*_\bullet$ and the cohomology with values in $\mathbf{IC}^*_\bullet$. Therefore, the sheaf $\mathbf{IN}^*_\bullet$ satisfies conditions (AX1) and, by [2, Theorem 2.5], there exists a quasi-isomorphism between $\mathbf{IN}^*_\bullet$ and $\mathbf{IC}^*_\bullet$ (see also [9]). As a consequence, these two sheaves have a common injective resolution and we may apply to it the uniqueness of cup_i-products established by M. Goresky in [7, Proposition 3.6]. □

We deduce from this study an isomorphism of algebras of cohomology, with coefficients in $\mathbb{F}_2$, by noting that the arguments can easily be adapted to any field of coefficients.

Corollary 4.4. *If X is a pseudomanifold, there are isomorphisms of perverse algebras*

$$H^*_\bullet(\underline{\mathrm{ISing}}^{\mathcal{F}}(X);\mathbb{F}_2) \cong H^*_\bullet(X;\mathbf{IN}) \cong H^*_\bullet(X;\mathbf{IC}).$$

Moreover, if X is compact and PL, one has also an isomorphism of algebras,

$$H^*_\bullet(X;\mathbf{IC}) \cong H^{\overline{t}-\overline{\bullet}}_{n-*}(X;\mathbb{F}_2),$$

with the intersection product on the last term.

Proof. The two first isomorphisms are consequences of the previous results on cup_i-products. The last one is established by Friedman in [6]. □

5. Pseudomanifolds with isolated singularities

In this section, we determine Steenrod squares on pseudomanifolds with isolated singularities. In this case, if the pseudomanifold is of dimension n, the perversity $\overline{p}$ is determined by one number, $\overline{p}(n)$. Recall now that the intersection cohomology of a cone cY on a space Y is given by $H^r_{\overline{p}}(cY) = H^r(Y)$, if $r \leq \overline{p}(n)$ and 0 otherwise.

Normal pseudomanifolds with isolated singularities are particular cases of the next result.

Proposition 5.1. *Let $\overline{p}$ be a loose perversity and M be a pseudomanifold obtained from a manifold with boundary, $(W,\partial W)$, by attaching cones on the connected components of the boundary, i.e., we have $\partial W = \sqcup_{u\in I}\partial_u W$, with $\partial_u W$ a connected component of ∂W, and M is the push out*

$$\begin{array}{ccc} \sqcup_{u\in I}\partial_u W & \xrightarrow{\iota} & W \\ \downarrow & & \downarrow \\ \sqcup_{u\in I} c(\partial_u W) & \longrightarrow & M. \end{array}$$

We filter the space M by $\emptyset \subset \{\vartheta_u \mid u \in I\} \subset M$, where ϑ_u is the summit of the cone on $\partial_u W$. Then, the following properties are statisfied.

(i) The cochain complex $\widetilde{N}^*_{\overline{p}}(M)$ is quasi-isomorphic to $N^*(W)\oplus_{N^*(\partial W)}\tau_{\leq\overline{p}(n)}N^*(\partial W)$, where $\tau_{\leq\overline{p}(n)}N^*(\partial W)$ is the usual truncation (see [2, Page 52]*).*

(ii) The intersection cohomology of M is determined by

$$H^k_{\overline{p}}(M;\mathbb{Z}) = \begin{cases} H^k(W;\mathbb{Z}) & \text{if } \; k \leq \overline{p}(n), \\ \operatorname{Ker} H^k(W;\mathbb{Z}) \to H^k(\partial W;\mathbb{Z}) & \text{if } \; k = \overline{p}(n)+1, \\ H^k(W,\partial W;\mathbb{Z}) & \text{if } \; k > \overline{p}(n)+1. \end{cases}$$

(iii) If $(\alpha,\iota^*\alpha) \in N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq \overline{p}(n)} N^*(\partial W)$ is a cocycle of $\overline{p}$-intersection and i is a positive integer, we have

$$\operatorname{Sq}^i(\alpha,\iota^*\alpha) = (\operatorname{Sq}^i\alpha, \iota^*\operatorname{Sq}^i\alpha) \in H^{*+i}_{\mathcal{L}(\overline{p},i)}(M;\mathbb{F}_2).$$

Proof. Starting from a triangulation of $(W,\partial W)$, we may suppose that M, W, ∂W and $c(\partial W) := \sqcup_{u\in I} c(\partial_u W)$ are triangulated in such a way that any simplex is filtered, for the filtration $\emptyset \subset \{\vartheta_u \mid u \in I\} \subset M$. Denote by $\underline{\mathcal{T}}_M$, $\underline{\mathcal{T}}_W$, $\underline{\mathcal{T}}_{\partial W}$, $\underline{\mathcal{T}}_{c(\partial W)}$ the associated filtered face sets. The canonical map

$$\widetilde{N}^*_{\overline{p}}(\underline{\operatorname{ISing}}^{\mathcal{F}}(M)) \to \widetilde{N}^*_{\overline{p}}(\underline{\mathcal{T}}_M)$$

is a quasi-isomorphism, see [12, Theorem 10] for instance. By construction of the triangulations, we have

$$\widetilde{N}^*_{\overline{p}}(\underline{\mathcal{T}}_M) = \widetilde{N}^*_{\overline{p}}(\underline{\mathcal{T}}_W) \oplus_{\widetilde{N}^*_{\overline{p}}(\underline{\mathcal{T}}_{\partial W})} \widetilde{N}^*_{\overline{p}}(\underline{\mathcal{T}}_{c(\partial W)}).$$

As M, W and ∂W are manifolds and $\widetilde{N}^*_{\overline{p}}(c\partial W)$ is quasi-isomorphic to a truncation of $N^*(\partial W)$ (see [8] or [2]), we have obtained a quasi-isomorphism between $\widetilde{N}^*_{\overline{p}}(\underline{\operatorname{ISing}}^{\mathcal{F}}(M))$ and $N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq\overline{p}(n)} N^*(\partial W)$. An element of this sum is of the type $(\alpha,\iota^*\alpha)$, with $\iota^*\alpha$ of degree less than, or equal to, $\overline{p}(n)$. This means that, if α is of degree k, we must have

$$\begin{cases} \iota^*\alpha = 0 & \text{if } \; k > \overline{p}(n), \\ \iota^*\alpha \text{ is a cocycle} & \text{if } \; k = \overline{p}(n), \\ \text{no condition} & \text{if } \; k < \overline{p}(n). \end{cases}$$

This implies immediately that $H^k_{\overline{p}}(M;\mathbb{Z}) = H^k(W;\mathbb{Z})$ if $k \leq \overline{p}(n)$ and that $H^k_{\overline{p}}(M;\mathbb{Z}) = H^k(W,\partial W;\mathbb{Z})$ if $k > \overline{p}(n)+1$. In degree $k = \overline{p}(n)+1$, the $\overline{p}$-intersection cohomology of M is formed of the elements of $H^k(W;\mathbb{Z})$ which are in the image of $H^k(W,\partial W;\mathbb{Z})$, i.e., that is the kernel of $H^k(W;\mathbb{Z}) \to H^k(\partial W;\mathbb{Z})$.

Finally, the quasi-isomorphism between $\widetilde{N}^*_{\overline{p}}(M)$ and $N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq\overline{p}(n)} N^*(\partial W)$ is a quasi-isomorphism of perverse DGA's ([4, Definition 10.3]), compatible with the cup$_i$-products, and the last complex can be used for the determination of cup$_i$-products, i.e., we have

$$(\alpha,\iota^*\alpha) \cup_i (\beta,\iota^*\beta) = (\alpha \cup_i \beta, \iota^*\alpha \cup_i \iota^*\beta),$$

from which we deduce the announced formula for Steenrod squares. □

Remark 5.2. Let $(\alpha,\iota^*\alpha)$ be a cocycle in $N^k(W) \oplus_{N^k(\partial W)} \tau_{\leq\overline{p}(n)} N^k(\partial W)$, , the perverse degree of the Steenrod square, $\operatorname{Sq}^j(\alpha,\iota^*\alpha) = (\alpha,\iota^*\alpha) \cup_{k-j} (\alpha,\iota^*\alpha)$, verifies

$$\|(\alpha,\iota^*\alpha) \cup_{k-j} (\alpha,\iota^*\alpha)\| \leq |\iota^*\alpha \cup_{k-j} \iota^*\alpha| \leq \overline{p}(n)+j.$$

This remark gives a direct proof of Goresky and Pardon's conjecture in the case of isolated singularities.

Finally, observe that the fact that the image of $H^*_{\overline{p}}$ by Sq^j is in perversity $\mathcal{L}(\overline{p},j) = \min(2\overline{p},\overline{p}+j)$ is perfectly in phase with the characterization of the intersection cohomology of M, made in *(ii)* of Proposition 5.1, and the definition of $\mathrm{Sq}^j(\alpha,\iota^*\alpha)$.

If N is a manifold, Proposition 5.1, applied to the product $N\times[0,1]$, gives back the well-known example of the suspension ΣN and proves that the classical formula of commutation between suspension and Steenrod squares, $\Sigma\circ\mathrm{Sq}^j = \mathrm{Sq}^j\circ\Sigma$, can be extended to this situation. We leave it to the reader and consider now the case of the Thom space of a vectorial bundle.

Example 5.3 (*Steenrod squares on the intersection cohomology of a Thom space*)**.** Let $\mathbb{R}^m\to E\to B$ be a vector bundle. We denote by $D_E\to B$ the associated disk-bundle and by $f\colon S_E\to B$ the associated sphere-bundle. The *Thom space,* $\mathrm{Th}(E)$, is built from the disk-bundle along the process described in Proposition 5.1; we filter it by the point of compactification.

Let $\overline{p}$ be a loose perversity. From Proposition 5.1, we recover (see [13, Page 77]) the intersection cohomology of the Thom space,

$$H^k_{\overline{p}}(\mathrm{Th}(E);\mathbb{F}_2) = \begin{cases} H^k(B;\mathbb{F}_2) & \text{if } k\leq\overline{p}(n),\\ \mathrm{Ker}\,(H^k(B;\mathbb{F}_2)\to H^k(S_E;\mathbb{F}_2)) & \text{if } k=\overline{p}(n)+1,\\ H^k(\mathrm{Th}(E);\mathbb{F}_2)\cong H^{k-m}(B;\mathbb{F}_2) & \text{if } k>\overline{p}(n)+1,\end{cases}$$

where the Thom's isomorphism, $H^k(\mathrm{Th}(E);\mathbb{F}_2)\cong H^{k-m}(B;\mathbb{F}_2)$, is given by the cup-product with the Thom class, θ, i.e., by $\gamma\mapsto\theta\cup f^*(\gamma)$. Observe also, from the Gysin sequence, that $\mathrm{Ker}\,(H^k(B;\mathbb{F}_2)\to H^k(S_E;\mathbb{F}_2))\cong\mathrm{Im}\,(-\cup e\colon H^{k-m}(B;\mathbb{F}_2)\to H^k(B;\mathbb{F}_2))$, where e is the Euler class.

• *In the case $k\leq\overline{p}(n)+1$, the Steenrod squares on* $H^*_{\overline{p}}(\mathrm{Th}(E);\mathbb{F}_2)$ coincide with the Steenrod squares on $H^*(B;\mathbb{F}_2)$.

• *For $k>\overline{p}(n)+1$,* the Cartan formula gives, with $\gamma\in H^*(B;\mathbb{F}_2)$,

$$\begin{aligned}\mathrm{Sq}^j(\theta\cup f^*(\gamma)) &= \sum_{\ell=0}^{j}\mathrm{Sq}^\ell(\theta)\cup\mathrm{Sq}^{j-\ell}(f^*(\gamma))\\ &= \sum_{\ell=0}^{j}\theta\cup f^*(\omega_\ell)\cup f^*(\mathrm{Sq}^{j-\ell}(\gamma)),\end{aligned}$$

where the ω_ℓ are the Stiefel-Whitney classes of f. If $\mu\in H^{m+k}(\mathrm{Th}(E));\mathbb{F}_2)$, there exists $\gamma\in H^k(B;\mathbb{F}_2)$ such that $\mu=\theta\cup f^*(\gamma)$. The Steenrod squares on $\mathrm{Th}(E)$, denoted by $\mathrm{Sq}_{\mathrm{Th}}$, and the Steenrod squares on B, denoted by Sq_B, are related by

$$\mathrm{Sq}^j_{\mathrm{Th}}(\mu)=\sum_{\ell=0}^{j}\theta\cup f^*(\omega_\ell)\cup f^*(\mathrm{Sq}_B^{j-\ell}(\gamma)).$$

More explicitly, if we denote by $\mathcal{T}h\colon H^k(\mathrm{Th}(E);\mathbb{F}_2)\to H^{k-m}(B;\mathbb{F}_2)$ the Thom isomorphism, we have

$$\mathcal{T}h\circ\mathrm{Sq}^j_{\mathrm{Th}}(-)=\sum_{\ell=0}^{j}\omega_\ell\cup(\mathrm{Sq}_B^{j-\ell}\circ\mathcal{T}h(-)).$$

6. Example of a fibration with fiber a cone

In this section, we construct an example showing the interest of the lifting of the image of Sq^i to the perversity $\mathcal{L}(\overline{p}, i)$ instead of $2\overline{p}$. As the case of Sq^1 was analyzed in [10], we choose an example with Sq^2.

Proposition 6.1. *There exists a pseudomanifold X and a GM-perversity $\overline{p}$, with an explicit non trivial perverse square,*

$$\mathrm{Sq}^2 \colon H^6_{\overline{p}}(X;\mathbb{F}_2) \to H^8_{\overline{p}}(X;\mathbb{F}_2),$$

whose composition with the canonical map $H^6_{\overline{p}}(X;\mathbb{F}_2) \to H^8_{2\overline{p}}(X;\mathbb{F}_2)$ is zero.

In this example, the domain and the range of the square Sq^2 own the same perversity $\overline{p}$, which shows that the general perversity of the range, $\mathcal{L}(\overline{p}, i)$, can sometimes be improved. But, as show the previous cases, the perversity $\mathcal{L}(\overline{p}, i)$ is optimal for a general statement.

Proof. The construction of the space X is done in two steps.

• First, we observe that the classifying map of the top class, $\mathbb{C}P(2) \times S^4 \to K(\mathbb{Z}, 8)$, lifts as a map $f \colon \mathbb{C}P(2) \times S^4 \to S^8$. We denote by $p_1 \colon E \to \mathbb{C}P(2) \times S^4$ the pullback of the Hopf fibration, $S^{15} \to S^8$, along f. We compose p_1 with the trivial fibration, $p_2 \colon \mathbb{C}P(2) \times S^4 \to \mathbb{C}P(2)$ and obtain a fibration

$$p \colon E \to \mathbb{C}P(2),$$

whose fiber, F, is $S^7 \times S^4$. To show this last point, consider the next commutative diagram:

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & E \arrow[r] \arrow[d, "p_1"] \arrow[dr, phantom, "\boxed{3}"] & S^{15} \arrow[d] \\
S^4 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{1}"] & \mathbb{C}P(2) \times S^4 \arrow[r, "f"] \arrow[d, "p_2"] & S^8 \\
* \arrow[r] & \mathbb{C}P(2) &
\end{tikzcd}$$

The rectangle formed of $\boxed{1}$ and $\boxed{2}$ is a pullback. As $\boxed{1}$ is a pullback, we deduce that $\boxed{2}$ is a pullback. Therefore, the rectangle formed of $\boxed{2}$ and $\boxed{3}$ is a pullback and the triviality of the map $S^4 \to S^8$ implies that F is $S^7 \times S^4$.

We study now the Serre spectral sequence of the fibration p. We denote by a_4, a_7 and $a_7 \cup a_4$ the generators of the reduced cohomology of $S^7 \times S^4$ and by x and x^2 the generators of the reduced cohomology of $\mathbb{C}P(2)$. An inspection of the degrees in the differentials, $d_r \colon E_r^{s,t} \to E_r^{s+r,t-r+1}$, shows that the only differential which can be potentially non trivial is

$$d_4 \colon E_4^{0,7} = E_2^{0,7} = \mathbb{F}_2 a_7 \to E_4^{4,4} = E_2^{4,4} = \mathbb{F}_2(x^2 \otimes a_4).$$

By definition of $S^7 \to E \to \mathbb{C}P(2) \times S^4$ as a pull-back of the Hopf fibration, we already know that the top class a_7 of S^7 transgress on the product $x^2 \cup a_4$. This gives $d_4(a_7) = x^2 \otimes a_4$ in the Serre spectral sequence of the fibration $p \colon E \to \mathbb{C}P(2)$.

We continue with the determination of the image of the cohomology class $x \otimes a_4$ by Sq^2 in $H^*(\mathbb{C}P(2); \mathbb{F}_2) \otimes H^*(S^7 \times S^4; \mathbb{F}_2)$. From the external Cartan formula, we have

$$\mathrm{Sq}^2(x \otimes a_4) = \mathrm{Sq}^2(x) \otimes a_4 + \mathrm{Sq}^1(x) \otimes \mathrm{Sq}^1(a_4) + x \otimes \mathrm{Sq}^2(a_4).$$

The last two terms are zero, for degree reasons. The equality $\mathrm{Sq}^2(x) = x^2$ gives

$$\mathrm{Sq}^2(x \otimes a_4) = x^2 \otimes a_4.$$

• The second step is the fibrewise conification, $c(S^7 \times S^4) \to X \xrightarrow{\hat{p}} \mathbb{C}P(2)$, of the fibration p. If $x \in \mathbb{C}P(2)$, we denote by $(S^7 \times S^4)_x$ the fiber over x and by ϑ_x the summit of the cone $c((S^7 \times S^4)_x)$. A continuous section μ of p, defined by $\mu(x) = \vartheta_x$, identifies $\mathbb{C}P^2$ to a closed subspace of X. We filter X by $\emptyset \subset X_0 = \mathbb{C}P(2) \subset X$. Observe that the singular set in X is $\mathbb{C}P(2)$ and that the link of a point of X_0 is $S^7 \times S^4$.

One can cover the space $\mathbb{C}P(2)$ with two open sets, U_1, U_2, such that U_1 is contractible, U_2 has the homotopy type of S^2 and $U_1 \cap U_2$ the homotopy type of S^3. The pullbacks of the fiber space X, over each of these three open sets, is trivial and we may use a Künneth formula for the determination of the $\overline{p}$-cohomology of the corresponding total spaces. Moreover, if $\overline{p}$ is a perversity, the cochain complex $\widetilde{N}^*_{\overline{p}}(c(S^7 \times S^4))$ is quasi-isomorphic to a sub-complex of $N^*(S^7 \times S^4)$. This implies that the cochain complex $\widetilde{N}^*_{\overline{p}}(X)$ is quasi-isomorphic to

$$(H^*(\mathbb{C}P(2); \mathbb{F}_2) \otimes H^{\leq \overline{p}(12)}(S^7 \times S^4; \mathbb{F}_2), \delta),$$

with $\delta(a_7) = x^2 \otimes a_4$, if $7 \leq \overline{p}(12)$, and $\delta = 0$ on the other elements. (We leave the details to the reader, quoting that this quasi-isomorphism can certainly also be deduced from a spectral sequence argument with [5, Theorem 3.5].)

In the case of a cone, there is only one stratum and the perversity $\overline{p}$ is determined by one number, which is $\overline{p}(12)$ in this example. If $\overline{p}(12) = k$, we denote the perversity $\overline{p}$ by $\overline{k}$. The square Sq^2, that we have previously determined, takes birth in the perversity $\overline{4}$ and we have

$$\mathrm{Sq}^2 \colon H^6_{\overline{4}}(X; \mathbb{F}_2) = \mathbb{F}_2(x \otimes a_4) \to H^8_{\overline{4}}(X; \mathbb{F}_2) = \mathbb{F}_2(x^2 \otimes a_4).$$

Observe that $\overline{6} = \mathcal{L}(\overline{4}, \overline{4} + 2)$ and that Sq^2 still survives as map from $H^6_{\overline{4}}$ to $H^8_{\overline{6}} = H^8_{\overline{4}}$. But, as $\overline{8} = 2 \times \overline{4}$ and $H^8_{\overline{8}}(X; \mathbb{F}_2) = 0$, this square Sq^2 disappears if we express it as a map from $H^6_{\overline{4}}$ to $H^8_{2 \times \overline{4}}$. □

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